THURSDAY, JUNE 11, 1903.

DIFFERENTIAL EQUATIONS.

A Treatise on Differential Equations. By Prof. A. R. Forsyth, Sc.D., LL.D., Math.D., F.R.S. Pp. xvi + 511. Third Edition. (London: Macmillan and Co., Ltd., 1903.) Price 14*s.*

THE value of this useful text-book has been increased by the inclusion in the third edition of important additional matter.

The principal additions are an account of Runge's method for the approximate numerical solution of ordinary differential equations, of Frobenius's method for the integration of linear equations in series, and of Jacobi's theory of multipliers.

The chief modifications of the matter treated in the earlier editions occur in the treatment of Lagrange's linear partial differential equation of the first order, in the discussion of the condition of integrability of a total differential equation, and in the treatment of Riccati's equation.

Of the above-mentioned subjects the one of greatest theoretic interest is probably the treatment of Lagrange's equation, whilst the most useful is Frobenius's method of integrating linear equations in series.

The theoretic interest of the treatment of Lagrange's equation arises from the fact that until Goursat published his "Leçons sur l'Intégration des Équations aux dérivées partielles du premier ordre" in 1891, the widely used rule for the solution of Lagrange's equation had not received adequate demonstration.[1]

If $u = a$, $v = b$ furnish values of z in terms of x, v which satisfy the equation

$$P\partial z/\partial x + Q\partial z/\partial y = R,$$

where P, Q, R are any functions of x, y, z; and if $\psi(x, y, z) = 0$ be any other integral, then the condition $\dfrac{\partial\,[\psi, u, v]}{[x, y, z]} = 0$ must be satisfied, *not necessarily identically, but in virtue of the relation between* x, y, z *given by* $\psi(x, y, z) = 0$. It is only when the above condition is satisfied identically that ψ is a function of u, v. In this case ψ is certainly included in the general integral. But it is possible to take a case of the general integral, and put it into a form in which the Jacobian does not vanish identically; *e.g.* if $x\partial z/\partial x + y\partial z/\partial y = z$, we may take $u \equiv y/x$, $v \equiv x/z$, $\psi \equiv y/x - x/z$ and the Jacobian vanishes identically; but if we put $\psi \equiv yz - x^2$, then the Jacobian $= -2\psi/(xz^2)$, which vanishes only when the relation between the variables is such as to make $\psi = 0$. Finally, it is possible to have singular integrals, which cannot be expressed in the form of the general integral at all. In this case, let $u = a$, $v = b$ be two integrals, and let $f(x,y,z) = 0$ be any other integral, then by elimination of y, z express $f(x,y,z) = 0$ in the form $\phi(x, u, v) = 0$.[2]

Then if D denote partial differentiation when x, u, v are the independent variables, it can be shown that $PD\phi(x, u, v)/Dx$ must vanish, not identically, but in virtue of the relation between x, y, z given by $f(x, y, z) = 0$. Prof. Forsyth proves that if P, Q, R are regular for values of x, y, z in the vicinity of any point on the integral $f(x, y, z) = 0$, then this integral is included in the general integral. Taking as an example the equation

$$(1 + \sqrt{z - x - y})\,\partial z/\partial x + \partial z/\partial y = 2,$$

we may take

$$u \equiv 2y - z,\ v \equiv y + 2\sqrt{z - x - y};$$

and $z = x + y$ is an integral not included in the general integral. In this case

$$\phi(x, u, v) \equiv (1 - \sqrt{1 + v - u - x})^2,$$

and $PD\phi(x, u, v)/Dx = -\sqrt{z - x - y}$, which vanishes when $z = x + y$. In this case it is at once seen that the coefficient $P \equiv 1 + \sqrt{z - x - y}$ is not regular in the vicinity of points on the integral $z = x + y$.

A similar point, arising out of the conditional vanishing of a Jacobian, comes up in connection with Art. 12. It is there proved that an ordinary differential equation of the first order and degree, with coefficients which are one-valued functions of the variables, has only one independent primitive.

As soon as the reader reaches the subject of singular solutions, he is forced to ask himself why the reasoning in Art. 12 is inapplicable. He wishes to have an explanation of the fact that the many-valuedness of the coefficients causes the reasoning to fail.

Suppose the equation is $2dy/dx + x + \sqrt{x^2 + 4y} = 0$. Two primitives of this are $c^2 + cx - y = 0$ and $x^2 + 4y = 0$. Their Jacobian is $2(x + 2c)$, which does not vanish identically, but conditionally, viz., at the point of contact of the envelope $x^2 + 4y = 0$ by the complete primitive $c^2 + cx - y = 0$.

The method of Frobenius for integrating linear differential equations in series is explained on pp. 235–249, and is applied to the solution of Bessel's equation. It is of a more general character than the special method applied to the same equation in chapter v.; and it exhibits the connection between the two solutions found by it. The connection between the two solutions obtained in chapter v. is difficult to perceive; and Frobenius's method has the advantage both in directness and simplicity. It is a valuable addition to the book.

Runge's method for the numerical solution of differential equations has suffered somewhat in the compression which the author has found necessary. Nevertheless, one cannot help regretting the omission to state the geometrical meaning of the expressions employed, and the connection of the method with Simpson's rule for the approximate evaluation of an integral. The student will probably be greatly perplexed as to the origin of the various quantities introduced and used in the investigation.

There are several difficulties in the discussion of the differential equation which is satisfied by the hypergeometric series in chapter vi. Although the subject cannot be properly dealt with without assuming a knowledge of the theory of functions, which is not to be expected of the majority of the readers of the book, yet there are some very obvious difficulties which could be removed by short explanations.

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[1] See Chrystal, *Transactions* of the Royal Society of Edinburgh, vol. xxxvi. part ii., p. 551 (1891).

[2] Or by eliminating z, x in the form $\psi(y, u, v) = 0$, or by eliminating x, y in the form $\chi(z, u, v) = 0$.

It is stated (Art. 122) that there is a linear relation between any three of the twenty-four integrals of the equation. The limitation that it is essential to consider only such groups of three integrals as have a common domain does not appear until we reach Art. 124, where it seems to contradict the statement in Art. 122.

The twenty-four integrals are divided into six groups of four each, and the members of each group of four are described as being equal. It should be pointed out that the members of each group of four fall into two pairs, that the members of one of these pairs are equivalent to one another, as they have the same domain; but they do not have the same domain as the members of the other pair (which are equivalent to one another). The four members of a group of four are equivalent to one another only in the domain common to them all. The integrals of one pair are to be regarded as continuations of the integrals of the other pair. From this it follows that in any linear relation between three of the integrals, it is not possible to replace any integral by another member of the group of four to which it belongs without examining whether the integrals appearing in the final relation have a common domain.

For example, relation No. (vi.), p. 219, viz.:—

$$Y_1 = M_5Y_5 + N_5Y_6$$

is intelligible if we take

$$Y_1 = F(\alpha, \beta, \gamma, x)$$
$$Y_5 = (1 - x)^{-\alpha} F\left(\alpha, \gamma - \beta, \alpha - \beta + 1, \frac{1}{1-x}\right)$$
$$Y_6 = (1 - x)^{-\beta} F\left(\beta, \gamma - \alpha, \beta - \alpha + 1, \frac{1}{1-x}\right)$$

because these integrals have a common domain. But it becomes meaningless if we replace

$$Y_5 \text{ by } x^{-\alpha} F\left(\alpha, \alpha - \gamma + 1, \alpha - \beta + 1, \frac{1}{x}\right),$$

which belongs to the same group of four integrals as that previously taken for Y_5; and if we replace

$$Y_6 \text{ by } x^{-\beta} F\left(\beta, \beta - \gamma + 1, \beta - \alpha + 1, \frac{1}{x}\right);$$

for Y_1, Y_5, Y_6 have now no common domain, except possibly points on the unit circle. This peculiarity had been noticed by Kummer in his memoir on the hypergeometric series. He held that even supposing we make the changes described above for Y_5 and Y_6, the equation should not be rejected as meaningless; for the two sides are now the expansions of the same function of x, one proceeding according to powers of x, and convergent inside the unit circle, the other proceeding according to powers of $\frac{1}{x}$ and convergent outside the unit circle; and he illustrated the subject by deducing from one side of one of the equations the expansion of tan x in powers of x, and from the other side of the equation its expansion in powers of $\frac{1}{x}$.

The whole subject received a thorough revision by Goursat (in the *Annales* de l'École Normale Supérieure, Sér. ii. t. x. 1881), who shows that in some cases the linear relations between the three integrals do not possess the same form throughout the whole of the plane of the complex variable. There still remains, however, for future researchers the discovery of an algebraic demonstration of such equations as the linear relation between

$$F(\alpha, \beta, \gamma, x), \quad x^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, x),$$

and

$$F(\alpha, \beta, \alpha+\beta-\gamma+1, 1-x),$$

series proceeding respectively according to integra powers of x, non-integral powers of x, and integra powers of $(1 - x)$, where, however, the last series cannot be expanded in integral powers of x.

The following details may be noticed:—

I. There is some obscurity in the explanation given in the note to Art. 25.[1]

If the system of curves $f(x,y,c) = 0$ have a node-locus let the node on the curve $f(x,y,a) = 0$ be given by $\xi = \phi(a)$, $\eta = \psi(a)$.

The node-locus will be found by eliminating a between the last two equations. The point to be explained is the reason for the appearance of this locus as a factor in the equation $\text{Disct}_c\ f(x,y,c) = 0$.

The coordinates of the node on the curve $f(x, y, a+\delta a = 0$ may be called $\xi + \delta\xi$, $\eta + \delta\eta$. Then the following equations hold:—$f(\xi, \eta, a) = 0$, $\frac{\partial f}{\partial \xi} = 0$, $\frac{\partial f}{\partial \eta} = 0$; and the equations which can be obtained from them by changing ξ, η, a into $\xi + \delta\xi$, $\eta + \delta\eta$, $a + \delta a$ respectively. Of this last set of three only the first is required, viz. $f(\xi + \delta\xi, \eta + \delta\eta, a + \delta a) = 0$. Neglecting quantities of the second order, and using the preceding equations, it follows that $\frac{\partial f}{\partial a}\delta a = 0$. Hence the values $\xi = \phi(a)$, $\eta = \psi(a)$ satisfy $\frac{\partial f}{\partial a} = 0$, as well as $f = 0$, and therefore the node-locus is a factor of $\text{Disct}_c\ f(x,y,c) = 0$.

II. The properties of the Schwarzian derivative (Art. 62) may be thrown into a more symmetric form, viz:—

$$\begin{aligned} \{s, x\}(dx)^2 &= -\{x, s\}(ds)^2 \\ \{s, x\}(dx)^2 &= \{s, y\}(dy)^2 + \{y, x\}(dx)^2 \\ &= \{s, y\}(dy)^2 + \{y, v\}(dv)^2 + \{v, x\}(dx)^2. \end{aligned}$$

III. In Art. 192, the argument may be stated thus:— It is given that

$$\frac{\partial\left[\frac{\partial F}{\partial \xi}, \frac{\partial F}{\partial \eta}\right]}{\partial[\xi, \eta]} = 0.$$

From this it follows that

$$\frac{\partial\left[\xi\frac{\partial F}{\partial \xi} + \eta\frac{\partial F}{\partial \eta} - F, \frac{\partial F}{\delta \xi}\right]}{\partial[\xi, \eta]} = 0.$$

Hence the equation of the tangent plane to the surface $z = F(x,y)$, viz. $z = x\frac{\partial F}{\partial \xi} + y\frac{\partial F}{\partial \eta} - \left(\xi\frac{\partial F}{\partial \xi} + \eta\frac{\partial F}{\partial \eta} - F\right)$ can, by putting $\frac{\partial F}{\partial \xi} = \lambda$, be expressed in the form $z = \lambda x + y\phi(\lambda) + \psi(\lambda)$, so that it is expressible in terms of a single arbitrary parameter λ. The quantities $\xi, \eta, \frac{\partial F}{\partial \xi}, \frac{\partial F}{\partial \eta}$ are not all functions of a single parameter.

IV. The solutions of Laplace's equation, which have been discovered since the second edition of this book was printed, and in which the author has himself borne an honourable part, are, if we except an example very near to the end of the book, not mentioned.

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[1] The word "discriminant-equation' in the fourth line should be "differential equation.'

THE MAGNITUDE OF THE PROTEINIC MOLECULE.

Die Grösse des Eiweissmoleküls. By Dr. F. N. Schulz. Pp. viii+106. (Jena: Gustav Fischer, 1903.) Price 2.50 marks.

THIS work is the second part of the author's "Studien zur Chemie der Eiweissstoffe"; the first part is entitled "Die Krystallisation von Eiweissstoffen und ihre Bedeutung für die Eiweisschemie," and is also published by Gustav Fischer.

The book is composed of five chapters. The first deals with elementary composition as a measure of the magnitude of the proteïnic molecule, and fills twenty-four pages. In it the author discusses firstly the ash of proteïns. This he divides into essential and non-essential parts, without predicating chemical essentiality of the former. He concludes that the ash is of no value for the purpose under consideration. He deals next with the sulphur, and shows that it can be used to give minimal values. It is pointed out how the difference in the ease of its elimination affects the results, and the methods of its determination are discussed.

In the second chapter the products of substitution are considered. This chapter contains fifty-three pages. Of the natural bodies oxyhæmoglobin and caseïn are the only ones lending themselves to calculation. Consideration of artificial products yields no figures of value at present. The substances resulting from association of acids and bases with proteïns are not as yet available for purposes of calculation. The same may be said of those of metals with proteïns, with the possible exception of Harnack's copper-albuminates. The author points out, however, that these substances need closer study.

In connection with these bodies the author diverges into a consideration of certain properties of colloids, and indicates that associations of colloids may simulate chemical compounds. He states emphatically that use of such words as *combination* and *compound,* in the case of certain proteïns and proteïnic derivatives, may be unwarranted:—

> "Eine Hauptaufgabe dieser Abhandlung war es gerade, dass gezeigt wird, dass bisher *keine zwingenden Gründe* vorliegen, um z. B. bei den Metallalbuminaten, oder später bei den Halogenalbuminaten, Verbindungen der Eiweissstoffe nach stöchiometrischen Gesetzen annehmen zu müssen."

The products of interaction of proteïns and halogens (especially iodine) are dealt with at some length. The absence of harmony in the results of different observers is shown, and the complexity of the process is pointed out. The conclusion is reached that these substances are not yet trustworthy for computational purposes.

The subject is regarded in the third chapter from the aspect of the products of hydrolysis, and it is found that no single compound is of use for the required calculation. The chapter contains nine pages

The fourth chapter, which consists of six pages, deals with physical methods, and chiefly with the cryoscopic one. The author has again to regard the results with suspicion, owing to the ash and the undefined nature of the substances. There is apparently an indication that the molecular masses of peptones, proteoses, and more complex proteïns stand to one another in a series of increasing magnitudes. The numbers attached to the two former classes may be of the right order; those connected with the latter are, however, valueless.

The final chapter, containing four pages, is devoted to conclusions. The author considers that the present state of the subject is very unsatisfactory, and that the molecular magnitudes of the more complex proteïns cannot be even given with approximate certainty. Selected minimal values, as those of Vaubel, lying for the more complex proteïns between 5000 and 15,000, can be made to give apparent harmony. But, if selection is not made, the result is very different.

The necessity of starting with crystalline bodies, and of improved methods is emphasised. The author also lays stress on the necessity of studying proteïns in their colloidal aspect, saying:—

> "Ich bin der Meinung, dass eine gründliche Erforschung der colloidalen Eigenschaften der Eiweisskörper, das Räthsel der Eiweisschemie eher aufklären wird, als eine detaillirte Untersuchung der Krystallinischen Eiweissspaltungsproducte."

He adopts throughout a position of impartial criticism, which is eminently sound. The results hitherto obtained have for him no great positive value at present; this he attributes to insufficient precision in the modes of investigation, although admitting that the cause may be inherent in the proteïnic nature.

Some might urge that publication is in these conditions premature. But in the present state of proteïnic chemistry such a pamphlet as this, permeated with sane criticism, and summarising what is known in a clear and agreeable manner, can only be of value. The just appreciation of the extreme importance of a study of the colloidal nature of proteïns is a main feature of the work.

It is a regrettable fact that no index of subject-matter is appended, although there is one of authors, and a table of contents.

F. ESCOMBE.

PHYSIOLOGICAL REPORTS.

Reports from the Laboratory of the Royal College of Physicians, Edinburgh. Edited by Sir John Batty Tuke, M.D., and D. Noël Paton, M.D. Vol. viii. (Edinburgh: Oliver and Boyd, 1903.)

THIS volume represents the work done in the laboratory in 1900 and 1901, and though a year late in its appearance is none the less welcome for that. Apart from one paper on the pollution of the Tyne Estuary, it is devoted to pathology and physiology.

Throughout there are records of the energy and helpfulness of the superintendent, Dr. Noël Paton, and no less than one-third of the articles are by him, either alone or in conjunction with others. Indeed, his